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Note on certain Coefficients appearing in the Algebraical Development of the Perturbative Function. By R. T. A. Innes.

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I.

We find a short algebraical development of the Perturbative Function in terms of the mean anomalies in Laplace's *Mécanique Céleste.* This was extended by Burckhardt and de Pontécoulant. Later, Le Verrier gave a development to the 7th. powers of the eccentricities and 6th. powers of the inclinations (both inclusive), and this was extended to the 8th. powers by Bouquet. Le Verrier gave extended tables and numerous precepts with his development (*Les Annales,* i. and x.), and in consequence it has been widely used.

A more methodical and general development was given by Newcomb (*Ast. Papers,* v., parts 1 and 4, 1895). Earlier, Newcomb had published a development in terms of the eccentric anomalies (*Ast. Papers,* iii.) which he used in deriving the periodic perturbations of the four inner planets, but it would seem that the further adoption of this development has been tacitly condemned by its illustrious author—it appears to be necessary to state this explicitly, as in a recent didactive treatise on this subject (I refer to Mons. H. Poincaré's *Leçons de Méc. Cél.*, ii., part 1, 1907) the author's references to Newcomb omit his later and more generally useful papers.

The earlier developments (including Le Verrier's) make use of the Laplace coefficients

$$b_{\frac{1}{2}}^{(i)},\ b_{\frac{3}{2}}^{(i)},\ b_{\frac{5}{2}}^{(i)},\ \text{etc. etc.,}$$

and of their derivatives of the first and higher orders. An inspection of Le Verrier's formulæ will show that the coefficients which actually appear are in a slightly different form, viz.

$$b_{\frac{1}{2}}^{(i)},\ ab_{\frac{3}{2}}^{(i)},\ a^2b_{\frac{5}{2}}^{(i)},\ \text{etc. etc.,}$$

and that the derivatives of these are there implicitly.

Finding the numerical values of the various Laplace coefficients (especially if a is over $\frac{1}{2}$ and accuracy is required) is a wearisome process, but when a is less than 0·75 the actual work required can be lessened by using tables by Runkle and others. Well-known recurrence formulæ exist between the various coefficients, but unfortunately their use leads to accumulating errors.* Le Verrier

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* This was pointed out by Le Verrier in one of his earliest papers (Paris, 1841) and is well known, yet we find Mons. H. Poincaré, in the work already cited, recommending their use. Of course the recurrence formulæ are *mathematically* exact, but they are too inaccurate for astronomical use, paradoxical as it sounds!

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introduced transformations of power-series so as to facilitate their calculation, and gave extensive tables (*Les Annales*, ii.), but even Le Verrier's figures will not give the 5th. place correctly if a is so large as 0·825 (Titan-Hyperion).

Besides the recurrence formulæ, Le Verrier gave (*Les Annales*, ii. pp. 108–9) several isolated relations between groups of the coefficients, such as

$$\left(4a\frac{d}{da} + 14a^2\frac{d^2}{da^2} + 8a^3\frac{d^3}{da^3} + a^4\frac{d^4}{da^4}\right)b_{\frac{1}{2}}^{(0)} = \left(2a\frac{d}{da} + 4a^2\frac{d^2}{da^2} + a^3\frac{d^3}{da^3}\right)b_{\frac{3}{2}}^{(1)}.$$

He found ten such relations, but gave no indication that a generalisation exists. The advantage of such relations is that they consist of simple additions of tabulated quantities multiplied by integers. The recurrence formulæ previously mentioned involve awkward multiplications by fractional numbers and by $\frac{1}{a} + a$, $1 + na^2$, etc.

II.

The object of this paper is to give a general formula which leads to a very simple recurrence formula, and embraces the generalisation mentioned above. This will be done for Newcomb's development. I will use the symbol introduced recently by Cowell—it has the great advantage of avoiding the use of an affix in the usual place of a power-index. The coefficients will compare as follows :—

Laplace.	Newcomb.	This paper.
$b_{\frac{1}{2}}^{(i)}$	$c_1^{(i)}$	$b_{i,1}$
$a\ b_{\frac{3}{2}}^{(i)}$	$c_3^{(i)}$	$b_{i,3}$
$a^2\ b_{\frac{5}{2}}^{(i)}$	$c_5^{(i)}$	$b_{i,5}$
etc.	etc.	etc.

In place of the ordinary derivatives, Newcomb has introduced logarithmic derivatives, indicated by the operator $D = a\frac{d}{da}$. The general relation is

$$\left(a\frac{d}{da}\right)^n = D(D-1)(D-2)\ \ldots\ .\ (D-n+1),$$

and it enables us to pass from one set of derivatives to the other if wanted.

The recurrence formula used by Newcomb

$$c_{n+2}^{(i,j)} = \frac{n+2j+na^2}{n^2a(1-a^2)}c_n^{(i,j+1)} + \frac{(n+2j)(2i+2-n)}{(n^21-a^2)}c_n^{(i+1,j)}$$

is free from some of the defects of that of Laplace. The quantity

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$c_n^{(i,j)}$ is an intermediary between $D^j c_n^{(i)}$ and $c_n^{(i)}$, which was introduced by Newcomb to facilitate calculation.* It is, to a simple factor, a hypergeometric series. If we apply the Gaussian relation

$$F_{\beta+} - F = \frac{\alpha}{\gamma}\, x F'''$$

to this quantity we get

$$(n-2)^2\, c_n^{(i+1,j)} = (2n+j-1)c_{n-2}^{(i,j+1)} + c_{n-2}^{(i,j+2)},$$

a recurrence formula freed from all defects. But what we really require are the quantities $c^{(i)}$, $Dc^{(i)}$, $D^2c^{(i)}$, etc. If we apply the formulæ given by Newcomb to find these quantities (*Ast. Papers*, v. p. 313, etc.) the last recurrence formula becomes

$$(n-2)^2 D^m c_n^{(i)} = D^m\left(D + i + \frac{n-3}{2}\right)\left(D - i - \frac{n-5}{2}\right)c_{n-2}^{(i-1)},$$

or using the newer notation

$$(n-2)^2 D^m b_{i,n} = D^m\left(D + i + \frac{n-3}{2}\right)\left(D - i - \frac{n-5}{2}\right)b_{i-1,n-2}.$$

Particular cases are—

$$\begin{aligned}
D^m c_3^{(0)} &= D^m(D^2 + D - 0)c_1^{(1)} \\
D^m c_3^{(1)} &= D^m(D^2 + D - 0)c_1^{(0)} = D^m(D^2 + D - 2)c_1^{(2)} \\
D^m c_3^{(2)} &= D^m(D^2 + D - 2)c_1^{(1)} = D^m(D^2 + D - 6)c_1^{(3)} \\
\cdots\quad & \quad\cdots \quad\quad \cdots \\
9D^m c_5^{(0)} &= D^m(D^2 + D - 0)c_3^{(1)} \\
9D^m c_5^{(1)} &= D^m(D^2 + D - 2)c_3^{(0)} = D^m(D^2 + D - 0)c_3^{(2)} \\
\cdots\quad & \quad\cdots \quad\quad \cdots \\
25D^m c_7^{(0)} &= D^m(D^2 + D - 2)c_5^{(1)} \\
25D^m c_7^{(1)} &= D^m(D^2 + D - 6)c_5^{(0)} = D^m(D^2 + D - 0)c_5^{(2)} \\
\cdots\quad & \quad\cdots \quad\quad \cdots \\
49D^m c_9^{(0)} &= D^m(D^2 + D - 6)c_7^{(1)} \qquad \text{etc. etc.}
\end{aligned}$$

With these new formulæ we may check some of the figures furnished by Newcomb (*Ast. Papers*, v. p. 343) for Venus-Earth. We have

$$\begin{array}{lrl}
 & D^4c_3^{(3)} = & 66839 \quad \text{(Newcomb)} \\
\hline
\text{Check} & D^6c_1^{(2)} = & 64691 \\
 & D^5c_1^{(2)} = & 4182 \\
 & -6D^4c_1^{(2)} = & -2033{\cdot}4 \\
\hline
\text{Hence} & & 66839{\cdot}6 = D^4c_3^{(3)}.
\end{array}$$

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* It was also used in the same sense by W. Scheibner (*Störungsfunction*, Gotha, 1853).

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If $D^4c_3^{(4)}$ is thus calculated, it is found $=65583$, whereas Newcomb has 65850, but an independent calculation has refound the former figure.

Again,

$$\begin{array}{lrcr}
 & Dc_9^{(4)} = & & 60000 \\
\hline
\text{Check} & D^3c_7^{(3)} = & & 2974000 \\
 & D^2 \;,, = & & 118490 \\
 & -42D \;,, = & & -224952 \\
\hline
 & 1/49 \times & & 2867538 = 58521 = Dc_9^{(4)}.
\end{array}$$

Newcomb (apparently the nearest round figure which is sufficiently accurate).

From the general formula many simple relations may be deduced, such as

$$D^m(b_{0,3} - b_{2,3}) = 2D^m b_{1,1}$$

$$3D^m(b_{0,5} - b_{2,5}) = 2D^m b_{1,3}$$

$$5D^m(b_{0,7} - b_{2,7}) = 2D^m b_{1,5} \qquad \text{etc. etc.}$$

The importance of the general formula in the development of the Perturbative Function is considerable. It not only simplifies the calculation of $b_{i,3}$, $b_{i,5}$, etc., but it can also be turned to use in computing the $D^m b_{i,1}$ coefficients, which will save the introduction of the $c^{(i,j)}$ quantities.

We have Newcomb (*Ast. Papers*, v. p. 310), and noting that

$$Dc_1^{(i)} = ic_1^{(i)} + c_1^{(i,1)}$$

$$\tfrac{1}{2}(D - i)b_{i,1} = \frac{1.3.5 \;.\;.\;.\; 2i+1}{2.4.6 \;.\;.\;.\; 2i+2}\alpha^{i+2}F(\tfrac{3}{2},\ \tfrac{3}{2}+i,\ 2+i,\ \alpha^2)$$

which transforms into

$$\tfrac{1}{2}(D - i)b_{i,1} = \frac{1.3.5 \;.\;.\;.\; 2i+1}{2.4.6 \;.\;.\;.\; 2i+2}\alpha^{i}\beta^2F(\tfrac{1}{2},\ \tfrac{1}{2}+i,\ 2+i,\ \alpha^2)$$

wherein

$$\beta^2 = \frac{\alpha^2}{1 - \alpha^2}.$$

Tisserand (*Méc. Cél.*, i. p. 274) shows that

$$b_{i,1} = \alpha^i \frac{2}{\pi}\int_0^\pi \frac{\sin^{2i}\theta\, d\theta}{\sqrt{1 - \alpha^2 \sin^2\theta}}$$

and we also have

$$F(\tfrac{1}{2},\ \tfrac{1}{2}+i,\ 2+i,\ \alpha^2) = N\frac{2}{\pi}\int_0^\pi \frac{\sin^{2i}\theta - \sin^{2i+2}\theta}{\sqrt{1 - \alpha^2\sin^2\theta}}d\theta,$$

wherein N is the ratio of two whole numbers.

Hence

$$(D - i)b_{i,1} = (2i + 1)\alpha^i\beta^2\left(\frac{b_{i,1}}{\alpha^i} - \frac{b_{i+1,1}}{\alpha^{i+1}}\right),$$

and $Db_{i,1}$ can be found from $b_{i,1}$ and $b_{i+1,1}$.

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Later, it will be seen that we should compute

$$p_i = \frac{b_{i,1}}{a^i},$$

and this must be done with precision to avoid the introduction of error.

To find the higher derivatives we can use the differential equation satisfied by the coefficients (Tisserand, *Méc. Cél.*, i. p. 281) which in the new notation is as follows—

$$(D^2 - i^2)b_{i,1} = \beta^2(2D + 1)b_{i,1},$$

and noting that

$$D\beta^2 = 2\beta^2(1 + \beta^2)$$

we have

$$\begin{aligned} D(D^2 - i^2)b_{i,1} &= \beta^2[4D^2 + 5D + (2 - 2i^2)]b_{i,1} \\ D^2(D^2 - i^2)b_{i,1} &= \beta^2[6D^3 + 13D^2 + (12 - 4i^2)D + (4 - 4i^2)]b_{i,1}, \\ &\text{etc.} \qquad \text{etc.} \end{aligned}$$

To obtain either $D^{m+2}b_{0,1}$ or $D^{m+2}b_{1,1}$ we must use these formulæ; for other values of i there is a better process. Between $D^m b_{0,1}$ and $D^m b_{1,1}$ we have the following simple relations—

$$\begin{aligned} Db_{0,1} &= aDb_{1,1} \\ D^2 b_{0,1} &= aD(D + 1)b_{1,1} \\ \cdot\ \cdot\ \cdot\ \cdot\ & \quad \cdot\ \cdot\ \cdot\ \cdot \\ D^m b_{0,1} &= aD(D + 1)^{m-1}b_{1,1}. \end{aligned}$$

To obtain $D^{m+2}b_{i,1}$ for other values of i we can write the formula—

$$D^m b_{i,n} = D^m\left[D^2 + D - \left(i - \frac{n-5}{2}\right)\left(i + \frac{n-3}{2}\right)\right]b_{i-1,n-2}$$

first with i positive and then with i negative. Equating the two values so found and confining our attention to the case of $n = 1$ yields

$$\begin{aligned} D^m(D^2 + D - 2)b_{2,1} &= D^m(D^2 + D - 0)b_{0,1} \\ D^m(D^2 + D - 6)b_{3,1} &= D^m(D^2 + D - 2)b_{1,1} \\ \text{etc.} \qquad & \qquad \text{etc.} \end{aligned}$$

so that in actual work we have

$$\begin{aligned} D^{m+2}b_{2,1} &= D^m(D^2 + D - 0)b_{0,1} - D^m(D - 2)b_{2,1} \\ D^{m+2}b_{3,1} &= D^m(D^2 + D - 2)b_{1,1} - D^m(D - 6)b_{3,1} \\ D^{m+2}b_{4,1} &= D^m(D^2 + D - 6)b_{2,1} - D^m(D - 12)b_{4,1} \\ \text{etc.} \qquad & \qquad \text{etc.} \end{aligned}$$

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Thus all the quantities except the $b_{i,1}$ can be derived quite simply once the latter are found, no large numbers nor awkward factors are involved, and there is no accumulation of error except that which is inevitable in any extended calculation, and which arises from the uncertainty of the last decimal place.

The formulæ developed above are useful in computing the action of the planets on the Moon, and evade a difficulty to which Brown has called attention (*Action of Planets*, p. 34).

It will be apparent that the introduction by Newcomb of the operator D instead of $\alpha\frac{d}{d\alpha}$ is not only an analytical improvement, but one of great practical importance as well.

I am now finishing a numerical application to the case of Titan-Hyperion, for which α exceeds 0·825, in which the whole process, with checks and alternative methods, is fully illustrated.

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Numerical Example. (Innes.)

Calculation of $b_{0,1}$.

This is the most important constant amongst the coefficients $D^m b_{i,n}$, as its value enters into all of them.

Earlier investigators computed its value by means of the hypergeometrical series $F(\frac{1}{2}, \frac{1}{2}, 1, \alpha^2)$, which is not very converging, especially for $\alpha > 0·707$. Le Verrier gave a transformation of this series (*Annales*, ii. p. [5]), which was used by Runkle in his well-known tables, but the transformation is still unsuited for a rapid and exact calculation, and in fact Runkle's 7th. decimal place is frequently in error; thus for $\alpha = 0·75$ Runkle gives

$$\log b_{01} = 0·3861679,$$

whereas the true value is $= 0·3861685$.

Newcomb, in his planetary theories (*Astr. Papers*, iii. p. 69), recommends a formula suggested by the use of Gauss's arithmetico-geometrical mean, viz.

$$\log b_{01} = \text{N} - \log \cos \tfrac{1}{2}\theta - \tfrac{1}{4} \log \cos \theta \qquad (\alpha = \sin \theta)$$

wherein N is very nearly equal to log 2. A small table of N is given by Newcomb, but it has evidently been computed by means of Runkle's table, as it suffers from the same errors. The value of N is probably equal to

$$\log 2 + 2 \log(1 - 2q^4 + 2q^{16} - \text{etc.})$$

wherein q is the quantity known as Jacobi's Nome.

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I have preferred to calculate the value of $b_{0,1}$ by means of the following formulæ, which have been adapted from Verhulst (*Fonctions Elliptiques*, Bruxelles, 1841, p. 256):—

$$b_{0,1}=\frac{8}{(1+\sqrt{a_1})^2}(1+2q^4+2q^{16}+\text{ etc.})^2$$

or

$$=\frac{8}{\left(\frac{1+\sqrt{a_1}}{2}+\sqrt[4]{\frac{(1+a_1)\sqrt{a_1}}{2}}\right)^2}(1+2q^{16}+2q^{64}+\text{ etc.})^2.$$

It seems inadvisable to replace a_1 by $\cos\theta$.

In the first of these two formulæ, the terms in q amount to less than one unit in the 8th. place of decimals if a does not exceed 0·35; in the second formula the terms in q will only amount to one unit in the 8th. place of decimals if a is as large as 0·998, a value much larger than is ever met with. In the case of Titan-Hyperion ($a=0{\cdot}825$), the use of the second formula, *neglecting* the q terms, introduces an error of 7 in the 19th. decimal place of the logarithm of $b_{0,1}$.

If we write

$$\log b_{0,1}=\log 8-2\log(1+\sqrt{a_1})+\mathrm{N}_1$$

the quantity N_1 may be taken from the table on p. 649.

The following values of log q for different values of a will afford a rough estimate of the q terms in $b_{0,1}$:—

a	$\log q$	a	$\log q$	a	$\log q$
$a=0{\cdot}05$	$\log q=6{\cdot}20$	$a=0{\cdot}35$	$\log q=7{\cdot}91$	$a=0{\cdot}65$	$\log q=8{\cdot}53$
0·15	7·15	0·45	8·15	0·75	8·71
0·25	7·61	0·55	8·35	0·85	8·90

(*N.B.*—10 has been added to log q.)

As a numerical example we compute $\log b_{0,1}$ for $a=0{\cdot}75$ by the approximate formula

$$\log b_{0,1}=\log 8-2\log\left[\frac{1+\sqrt{a_1}}{2}+\sqrt[4]{\frac{(1+a_1)\sqrt{a_1}}{2}}\right].$$

$$a=\frac{3}{4}\quad.\quad a_1=\frac{\sqrt{7}}{4}\quad.\;.\;.\;.\quad\log a_1=9{\cdot}82048903$$

$$\frac{1+\sqrt{a_1}}{2}=0{\cdot}90664414\qquad\sqrt[4]{\frac{(1+a_1)\sqrt{a_1}}{2}}=0{\cdot}90661866$$

$$\log 8=0{\cdot}90308999$$

$$2\log 1{\cdot}81326280=0{\cdot}51692150$$

$$\log b_{0,1}=0{\cdot}38616849$$

It will be noticed that although q has been introduced into the formulæ, it is in reality not used (owing to the extraordinarily close

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approximation to the true value given by the second formula); thus it does not seem necessary to repeat the equations by which q may be found from α; they will be found on p. 494 of the *M.N.*, vol. lxii., where, however, k_1 is written for α_1.

The next step is to calculate $b_{i,1}$ from $i = 1$ to such a value of i as experience may indicate. Having found $b_{0,1}$, there is no better way of finding the other values than that first shown by Hansen, and more fully and simply set out by Newcomb (*Ast. Papers*, iii. p. 64). The process here followed only deviates in unessentials from that of Newcomb. I prefer to write

$$\frac{b_{i,1}}{\alpha^i} = b_{0,1}\, p_1\, p_2\, p_3 \; . \; . \; . \; . \; p_i,$$

because the quantities p change slowly for considerable changes in α. At first sight the computation of these quantities appears complicated, but on closer acquaintance it is found most simple. Once it is started, the only table required is Zech's subtraction-table, and in general only one opening of that. The one disadvantage is that Zech's table has at most only 7 decimal places, and no more extended table is available; hence it is not possible to be certain of the last figure; in spite of this, the exactitude is quite sufficient in practice.

The quantities $b_{0,1}$ and p_1, p_2, ... p_i are computed so quickly and easily that tables giving them with α as argument seem to be superfluous. To exhibit, however, the march of these quantities, Mr. Frank Robbins has computed the short table on p. 650. Short as this table is, it will permit the p_i to be taken out almost correctly by inspection.

In the case of Titan-Hyperion, for which $\alpha = 0{\cdot}8250634$ (H. Struve), the following are the chief figures involved—

$$\log \alpha = 9{\cdot}91648732$$

$$1 - \alpha^2 = \alpha_1^2 = 0{\cdot}31927039 \qquad \log = 9{\cdot}50415864$$

$$1 + \sqrt{\alpha_1} \quad = 1{\cdot}75169154 \qquad \log = 0{\cdot}24345763$$

Hence by the formulæ on p. 639,

$$b_{0,1} = 2{\cdot}6074655 \qquad \log = 0{\cdot}41621857$$

To calculate p_{11} use is made of the well-known Gaussian continued-fraction. In this calculation the letters a, b, c, d . . . . and F are simple fractional factors; Zech is put against the figures for which Zech's subtraction-table is used. The calculation is quite similar to the example given by Newcomb (*Ast. Papers*, iii. p. 71), but is more extended because of the large value of α.

co-log		− 0·34	− 1·486	− 0·2803	− 1·75292	− 0·231014	− 2·5051152
Zech		0·265	0·0144	0·32199	0·007740	0·3845441	0·0013594
f	9·83	e 8·416	d 9·8716	c 8·09211	b 9·928271	a 7·2773661	F 9·9797966
α^2	9·83	9·833	9·8330	9·83297	9·832975	9·8329746	
	9·66 −	8·514 −	9·7197 −	8·24708 −	9·768986 −	7·4948848 −	9·9811560 = p_{11}

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With this value of p_{11} we proceed

$$\begin{array}{ll} p_{11} \quad = & 9{\cdot}9811560 \\ a^2/(1+a^2) & 9{\cdot}6074768 \\ \text{Constant} & 0{\cdot}0211893 \\ \cline{2-2} & 9{\cdot}6098221 \\ \cline{2-2} \text{co-log} & 0{\cdot}3901779 \\ \cline{2-2} \text{Zech} & 0{\cdot}2271017 \\ \text{Constant} & 9{\cdot}9777236 \\ 1/(1+a^2) & 9{\cdot}7745021 \\ \cline{2-2} p_{10} & 9{\cdot}9793274 \end{array}$$

and so on; the only tabular figure in each block being taken from Zech.

The following values of p result—

$$\begin{array}{rl} \log p_1 = & 9{\cdot}7560111 \\ p_2 & 9{\cdot}8960884 \\ p_3 & 9{\cdot}9322424 \\ p_4 & 9{\cdot}9493080 \\ p_5 & 9{\cdot}9593536 \\ p_6 & 9{\cdot}9660080 \\ p_7 & 9{\cdot}9707556 \\ p_8 & 9{\cdot}9743201 \\ p_9 & 9{\cdot}9770987 \\ p_{10} & 9{\cdot}9793274 \\ p_{11} & 9{\cdot}9811560 \end{array}$$

In possession of these numbers we proceed to find

$$b_{1,1} = ab_{0,1}\,p_1$$
$$b_{2,1} = ab_{1,1}\,p_2 \qquad \text{etc. etc.,}$$

then $Db_{i,1}$ by

$$(D-0)b_{0,1} = \beta^2 b_{0,1}(1-p_1) \qquad \beta^2 = \left(\frac{a}{a_1}\right)^2$$
$$(D-1)b_{1,1} = 3\beta^2 b_{1,1}(1-p_2)$$
$$(D-2)b_{2,1} = 5\beta^2 b_{2,1}(1-p_3)$$
$$\text{etc.} \qquad\qquad \text{etc.}$$

We have now to find $D^{2+m}b_{0,1}$ and $D^{2+m}b_{1,1}$, which do not fall into a more simple general formula for $D^{2+m}b_{i+2,1}$. We therefore use

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$$D^0(D^2-i^2)b_{i,1}=\beta^2[2D+1]b_{i,1}$$ *N.B.*—$b_{i,1}$ is omitted in the next lines.

$$\begin{aligned}
D(D^2-i^2) &= \beta^2[4D^2+5D+2(1-i^2)]\\
D^2(D^2-i^2) &= \beta^2[6D^3+13D^2+(12-4i^2)D+4(1-i^2)]\\
D^3(D^2-i^2) &= \beta^2[8D^4+25D^3+(38-6i^2)D^2+(28-12i^2)D+8(1-i^2)]\\
D^4(D^2-i^2) &= \beta^2[10D^5+41D^4+(88-8i^2)D^3+(104-24i^2)D^2\\
&\qquad +(64-32i^2)D+16(1-i^2)]\\
D^5(D^2-i^2) &= \beta^2[12D^6+61D^5+(170-10i^2)D^4+(280-40i^2)D^3\\
&\qquad +(272-80i^2)D^2+(144-80i^2)D+32(1-i^2)]\\
D^6(D^2-i^2) &= \beta^2[14D^7+85D^6+(292-12i^2)D^5+(620-60i^2)D^4\\
&\qquad +(832-160i^2)D^3+(688-240i^2)D^2+(320-192i^2)D+64(1-i^2)].
\end{aligned}$$

Having found $D^m b_{0,1}$ and $D^m b_{1,1}$ by these formulæ we may check the values by means of the following equations:—

$$\begin{aligned}
Db_{0,1} &= \alpha Db_{1,1}\\
D(D-1)b_{0,1} &= \alpha D^2 b_{1,1}\\
\ldots\quad & \ldots\quad \ldots\\
D(D-1)^m b_{0,1} &= \alpha D^{m+1} b_{1,1}.
\end{aligned}$$

When satisfied as to the accuracy of the values obtained for these coefficients, we proceed with

$$\begin{aligned}
D^{m+2}b_{2,1} &= D^m[(D^2+D-0)b_{0,1}-(D-2)b_{2,1}]\\
D^{m+2}b_{3,1} &= D^m[(D^2+D-2)b_{1,1}-(D-6)b_{3,1}]\\
\ldots\quad & \ldots\quad \ldots\\
D^{m+i}b_{i,1} &= D^m[(D^2+D-(i-1)(i-2))b_{i-2,i}-(D-i(i-1))b_{i,1}].
\end{aligned}$$

With the above sets of formulæ the following table has been computed:—

Titan-Hyperion ($\alpha=0{\cdot}8250634$).

i	$b_{i,1}$	$Db_{i,1}$	$D^2b_{i,1}$	$D^3b_{i,1}$	$D^4b_{i,1}$	$D^5b_{i,1}$	$D^6b_{i,1}$
0	2·607466	2·389585	15·74935	170·9130	2706·378	56736·48	1482263
1	1·226640	2·896245	16·19242	171·8703	2713·127	56814·88	1483538
2	0·796696	2·820306	16·91202	175·3909	2735·725	57057·92	1487413
3	0·562370	2·611792	17·39780	180·5431	2776·454	57491·06	1494094
4	0·412874	2·359360	17·54728	186·1462	2834·065	58140·99	1503924
5	0·310212	2·098800	17·36659	191·2088	2904·347	59020·83	1517334
6	0·236676	1·847254	16·90218	195·0217	2981·309	60121·47	1534702
7	0·182556	1·613119	16·21328	197·1492	3058·366	61410·81	1556265
8	0·141972	1·400109	15·35940	197·3859	3129·180	62836·30	1582006
9	0·111120	1·209330	14·39452	195·7050	3188·272	64331·31	1611631
10	0·087419	1·040391					
11	0·069063						
12	(0·05478 est.)						
13	(0·04360 est.)						

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For values of $D^m b_{i,\,n}$ (n other than 1) the formulæ

$$n^2 D^m b_{i+1,\,n+2} = D^m\left[D^2 + D - \left(i + \frac{n-1}{2}\right)\left(i + \frac{n+1}{2}\right)\right] b_{i,\,n}$$

$$n^2 D^m b_{i-1,\,n+2} = D^m\left[D^2 + D - \left(i - \frac{n-1}{2}\right)\left(i - \frac{n+1}{2}\right)\right] b_{i,\,n}$$

will be used, firstly with $n = 1$, then $n = 3, 5, 7$, and so on.

The following empirical formula, which is only valid for large values of a, is curious—

$$D^6 b_{1,\,1} = D^6 b_{0,\,1} + \tfrac{1}{2}[D^4 - D^3 + D^2] b_{0,\,1}.$$

In the case of Titan-Hyperion this shows no deviation to 7 places of figures.

It will be remarked that from first to last no term of an infinite series is computed; the nearest approach to an infinite series is the continued fraction for p_{11}.

It is well to have checks of an entirely independent nature on the values of the coefficients. Suitable checks will be found in the following formulæ, where indeed use is made of infinite series, but these series are so convergent that even with a so large as 0·825 . . . three or four terms are sufficient to ensure accuracy in the eighth significant figure.

These formulæ have been derived by Gaussian transformations of the ordinary hypergometric series.

Put

$$a = \sin\theta, \qquad a_1 = \cos\theta, \qquad \frac{a}{a_1} = \tan\theta,$$

$$\cos^2\frac{\theta}{2} = \frac{1 + a_1}{2},$$

and to abbreviate

$$\sin\theta \tan^2\theta = \mathrm{st} \quad \text{and} \quad \tan^4\frac{\theta}{2} = x\,;$$

then

$$b_{0,\,1} = \quad 2\frac{1}{\cos^2\frac{\theta}{2}} F(\tfrac{1}{2}, \tfrac{1}{2}, 1.\, x)$$

$$(D + 1) b_{0,\,1} = \quad 1.2\frac{\cos^2\frac{\theta}{2}}{a_1^2} F(-\tfrac{1}{2}, -\tfrac{1}{2}, 1.\, x)$$

$$(D + 1)(D + 3) b_{0,\,1} = \quad 1.3.2\frac{\cos^6\frac{\theta}{2}}{a_1^4} F(-\tfrac{3}{2}, -\tfrac{3}{2}, 1.\, x)$$

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$$(\mathrm{D}+1)(\mathrm{D}+3)(\mathrm{D}+5)b_{0,1} = 1.3.5.2\frac{\cos^{10}\frac{\theta}{2}}{\alpha_1^6}\mathrm{F}(-\tfrac{5}{2},\ -\tfrac{5}{2},\ 1.\ x)$$

$$(\mathrm{D}+1)(\mathrm{D}+3)\ldots(\mathrm{D}+7)b_{0,1} = 1.3.5.7.2\frac{\cos^{14}\frac{\theta}{2}}{\alpha_1^8}\mathrm{F}(-\tfrac{7}{2},\ -\tfrac{7}{2},\ 1.\ x)$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$(\mathrm{D}-1)b_{1,1} = \frac{1.3}{2.4}.2\ \mathrm{st}\frac{1}{\cos^2\frac{\theta}{2}}\mathrm{F}(-\tfrac{1}{2},\ \tfrac{1}{2},\ 2.\ x)$$

$$(\mathrm{D}+2)(\mathrm{D}-1)b_{1,1} = \frac{3.5}{2.4}.2\ \mathrm{st}\frac{\cos^2\frac{\theta}{2}}{\alpha_1^2}\mathrm{F}(-\tfrac{3}{2},\ -\tfrac{1}{2},\ 2.\ x)$$

$$(\mathrm{D}+2)(\mathrm{D}+4)(\mathrm{D}-1)b_{1,1} = \frac{5.7}{2.4}\ 2\ \mathrm{st}\frac{\cos^6\frac{\theta}{2}}{\alpha_1^4}\mathrm{F}(-\tfrac{5}{2},\ -\tfrac{3}{2},\ 2.\ x)$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$(\mathrm{D}-2)(\mathrm{D}-4)b_{2,1} = 1.3.\frac{1.3.5.7}{2.4.6.8}\ 2\ (\mathrm{st})^2\frac{1}{\cos^2\frac{\theta}{2}}\mathrm{F}(-\tfrac{3}{2},\ \tfrac{1}{2},\ 3.\ x)$$

$$(\mathrm{D}+3)(\mathrm{D}-2)(\mathrm{D}-4)b_{2,1} = 1.3.\frac{3.5.7.9}{2.4.6.8}\ 2\ (\mathrm{st})^2\frac{\cos^2\frac{\theta}{2}}{\alpha_1^2}\mathrm{F}(-\tfrac{5}{2},\ -\tfrac{1}{2},\ 3.\ x)$$

$$(\mathrm{D}+3)(\mathrm{D}+5)(\mathrm{D}-2)(\mathrm{D}-4)b_{2,1} = 1.\ \frac{5.7.9.11}{2.4.6.8}\ 2\ (\mathrm{st})^2\frac{\cos^6\frac{\theta}{2}}{\alpha_1^4}\mathrm{F}(-\tfrac{7}{2},\ -\tfrac{3}{2},\ 3.\ x)$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$(\mathrm{D}-3)(\mathrm{D}-5)(\mathrm{D}-7)b_{3,1} = 1.3.5.\frac{1.3.5.7.\ 9\ .11}{2.4.6.8.10.12}\ 2\ (\mathrm{st})^3\frac{1}{\cos^2\frac{\theta}{2}}\mathrm{F}(-\tfrac{5}{2},\ \tfrac{1}{2},\ 4.\ x)$$

$$(\mathrm{D}+4)(\mathrm{D}-3)(\mathrm{D}-5)(\mathrm{D}-7)b_{3,1} = 1.3.5.\ \frac{3.5\ \ldots\ldots\ 13}{2.4\ \ldots\ldots\ 12}\ 2\ (\mathrm{st})^3\frac{\cos^2\frac{\theta}{2}}{\alpha_1^2}\mathrm{F}(-\tfrac{7}{2},\ -\tfrac{1}{2},\ 4.\ x)$$

$$(\mathrm{D}+4)(\mathrm{D}+6)(\mathrm{D}-3)\ldots(\mathrm{D}-7)b_{3,1} = 1.5\ \frac{5.7\ \ldots\ldots\ 15}{2.4\ \ldots\ldots\ 12}\ 2\ (\mathrm{st})^3\frac{\cos^6\frac{\theta}{2}}{\alpha_1^4}\mathrm{F}(-\tfrac{9}{2},\ -\tfrac{1}{2},\ 4.\ x)$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$(\mathrm{D}-4)(\mathrm{D}-6)(\mathrm{D}-8)(\mathrm{D}-10)b_{4,1} = 1.3.5.7\frac{1.3.5\ \ldots\ 15}{2.4.6\ \ldots\ 16}\ 2\ (\mathrm{st})^4\frac{1}{\cos^2\frac{\theta}{2}}\mathrm{F}(-\tfrac{7}{2},\ \tfrac{1}{2},\ 5.\ x)$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

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$$(\mathrm{D}-5)(\mathrm{D}-7)\ldots\ldots(\mathrm{D}-13)b_{5,1}=1.3.5.7.9\frac{1.3.5\ldots19}{2.4.6\ldots20}\,2\,(st)^5\,\frac{1}{\cos^2\frac{\theta}{2}}\mathrm{F}(-\tfrac{9}{2},\tfrac{1}{2},6.x)$$

$$\ldots\qquad\ldots\qquad\ldots\qquad\ldots$$

$$(\mathrm{D}-6)(\mathrm{D}-8)\ldots\ldots(\mathrm{D}-16)b_{6,1}=1.3.5\ldots11\,\frac{1.3.5\ldots23}{2.4.6\ldots24}\,2\,(st)^6\,\frac{1}{\cos^2\frac{\theta}{2}}\mathrm{F}(-\tfrac{11}{2},\tfrac{1}{2},7.x)$$

etc. etc. etc.

We also have by direct differentiation of $b_{0,1}$

$$\mathrm{D}b_{0,1}=\tan\theta\tan\frac{\theta}{2}\left[b_{0,1}+\frac{\tan^2\frac{\theta}{2}}{\cos\theta}\,\mathrm{F}(\tfrac{1}{2},\tfrac{1}{2},2.x)\right].$$

The formula for $b_{0,1}$ is only inserted in the above table for sake of symmetry. It can be further transformed, as is well known, but the process used for its further transformation will not apply to the other formulæ.

Special formulæ such as $(\mathrm{D}+2)(\mathrm{D}-1)\,b_{1,1}=(\mathrm{D}+3)(\mathrm{D}-2)\,b_{3,1}$ also exist.

If we apply some of these formulæ to the case of Titan-Hyperion we have

$$\theta=55^\circ\!\cdot594922$$

$$\log\tan^2\theta=0\cdot32881604.\quad\log\tan\frac{\theta}{2}=9\cdot72196186.\quad\log\cos^2\frac{\theta}{2}=9\cdot89349547$$

$$(\mathrm{D}-1)b_{1,1}=\frac{3}{4}\sin\theta\tan^2\theta\frac{1}{\cos^2\frac{\theta}{2}}\left(1-\frac{1}{8}x-\frac{1}{64}x^2-\frac{5}{1024}x^3-\ldots\ldots\right)$$

log series $=9\cdot99574456$

,, factors $=0\cdot22686915$

$(\mathrm{D}-1)b_{1,1}=0\cdot22261371=1\cdot6696049$

The previous calculation gives 1·6696046

Again,

$$(\mathrm{D}^3-15\mathrm{D}^2+71\mathrm{D}-105)b_{3,1}=\frac{3465}{512}\,\frac{(\sin\theta\tan^2\theta)^3}{\cos^2\frac{\theta}{2}}\left(1-\frac{5}{16}x+\frac{9}{128}x^2-\frac{5}{1024}x^3-\ldots\right)$$

log series $=9\cdot9895742$

,, factors $=0\cdot6728478$

$1\cdot6624220=45\cdot96434$

The tables give 45·96443

With these coefficients we could proceed to form a theory of the motion of Hyperion as disturbed by Titan, but it is at least doubtful if the algebraical development of the perturbing function

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converges in this case. K. F. Sundman, in considering movement in a plane (which is very nearly the case with Titan-Hyperion), shows that the development according to powers of the eccentricities is certainly divergent if

$$a(1 + 2e + e^2 + \text{etc.}) > (1 - 2e_1 + e_1^2 - \tfrac{2}{3}e_1^3 + \tfrac{2}{3}e_1^4 - \text{etc.})$$

(*Ueber* die Störungen . . . . Helsingfors, 1901).

In the present case we have

$$a = 0{\cdot}825$$

$$e = 0{\cdot}029$$

$$e_1 = 0{\cdot}104$$

so that the inequality is

$$0{\cdot}874 > 0{\cdot}8016$$

and the development is divergent. In this case e_1 is not really an eccentricity in the planetary sense of the word; it is a perturbation due to the action of Titan; but this probably does not invalidate Sundman's criterion.

Newcomb and H. Struve have computed several of the coefficients (*Ast. Papers*, iii., and *Poulkovo*, xi$_2$. pp. 267 and 283). Using H. Struve's value of a (0·8250634) and Newcomb's clear notation we have

$$k_1 = e_1\mathrm{P}_{0,\,1}^{0,\,1} + e_1^3\mathrm{P}_{0,\,1}^{0,\,3} + e_1^5\mathrm{P}_{0,\,1}^{0,\,5} + \text{etc.}$$

$$\mathrm{P}_{0,\,1}^{0,\,1} = \tfrac{1}{2}(7 + \mathrm{D})b_{3,\,1} = +3{\cdot}274193.$$

$$\mathrm{P}_{0,\,1}^{0,\,3} = \tfrac{1}{16}(-358 - 37\mathrm{D} + 10\mathrm{D}^2 + \mathrm{D}^3)b_{3,\,1} = +3{\cdot}534766.$$

$$\mathrm{P}_{0,\,1}^{0,\,5} = \tfrac{1}{384}(16072 + 286\mathrm{D} - 942\mathrm{D}^2 + 15\mathrm{D}^3 + 17\mathrm{D}^4 + \mathrm{D}^5)b_{3,\,1} = +262{\cdot}4885$$

$$k_2 = e_1^2\mathrm{P}_{0,\,2}^{0,\,2} + e_1^4\mathrm{P}_{0,\,2}^{0,\,4} + \text{etc.}$$

$$\mathrm{P}_{0,\,2}^{0,\,2} = \tfrac{1}{8}(202 + 29\mathrm{D} + \mathrm{D}^2)b_{6,\,1} = +14{\cdot}78513.$$

$$\mathrm{P}_{0,\,2}^{0,\,4} = \tfrac{1}{96}(-41572 - 5360\mathrm{D} + 113\mathrm{D}^2 + 34\mathrm{D}^3 + \mathrm{D}^4)b_{6,\,1} = -112{\cdot}1.$$

The value of $\mathrm{P}_{0,\,2}^{0,\,4}$ is derived from Newcomb's formulæ; those of Le Verrier give -5062 as the coefficient of $\mathrm{D}b_{6,\,1}$.

$$k_3 = e_1^3\mathrm{P}_{0,\,3}^{0,\,3} + \text{etc.}$$

$$\mathrm{P}_{0,\,3}^{0,\,3} = \tfrac{1}{48}(9846 + 1415\mathrm{D} + 66\mathrm{D}^2 + \mathrm{D}^3)b_{9,\,1} = +82{\cdot}31306.$$

. . . .

$$\mathrm{P}_0 = \tfrac{1}{2}(\mathrm{D} + \mathrm{D}^2)b_{0,\,1} = +9{\cdot}069466.$$

$$\mathrm{P}_1 = \tfrac{1}{2}(7 + \mathrm{D})(-36 + \mathrm{D} + \mathrm{D}^2)b_{3,\,1} = \tfrac{1}{2}(-63 - 2\mathrm{D} + 8\mathrm{D}^2 + \mathrm{D}^3)b_{3,\,1} = -139{\cdot}5363.$$

N.B. $\tfrac{1}{2}(b_{i-1,\,3} + b_{i+1,\,3}) = (\mathrm{D}^2 + \mathrm{D} - i^2)b_{i,\,1}.$

$$\mathrm{P}_2 = \tfrac{1}{8}(202 + 29\mathrm{D} + \mathrm{D}^2)(-36 + \mathrm{D} + \mathrm{D}^2)b_{6,\,1} = \tfrac{1}{8}(-7272 - 842\mathrm{D} + 195\mathrm{D}^2 + 30\mathrm{D}^3 + \mathrm{D}^4)b_{6,\,1} = +1106{\cdot}424.$$

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It will be noticed that the same coefficients repeat themselves in the last two quantities, but they are now operated on by (D^2+D-36).

Lastly, the values of the coefficients may be required for a slightly different value of a, say $a+\Delta a$. If so, the figures given for a will be useful. Let us represent any coefficient $D^m b_{i,n}$ by $f(a)$, then by Taylor's series

$$f(a+\Delta a)=f(a)+\frac{\Delta a}{a}a\frac{d}{da}f(a)+\tfrac{1}{2}\left(\frac{\Delta a}{a}\right)^2 a^2\frac{d^2}{da^2}f(a)+\text{ etc.}$$

In this replace the operator $a\dfrac{d}{da}$ by D which gives

$$a^2\frac{d^2}{da^2}=D(D-1)$$

$$a^3\frac{d^3}{da^3}=D(D-1)(D-2),$$

$$\text{etc.} \qquad \text{etc.}$$

and Taylor's series becomes

$$f(a+\Delta a)=f(a)+\log\left(1+\frac{\Delta a}{a}\right)Df(a)+\tfrac{1}{2}\left[\log\left(1+\frac{\Delta a}{a}\right)\right]^2 D^2 f(a)+\text{etc}$$

If $\dfrac{\Delta a}{a}$ is so small that its square is negligible we have

$$\log\left(1+\frac{\Delta a}{a}\right)=\frac{\Delta a}{a}.$$

If the Newcomb coefficient $D^m b_{i,n}$ be put in place of $f(a)$ we have putting l for $\log\left(1+\dfrac{\Delta a}{a}\right)$,

$$\overbrace{D^m b_{i,n}}^{\text{For } a+\Delta a} = \overbrace{D^m b_{i,n}+lD^{m+1}b_{i,n}+2!\,l^2D^{m+2}b_{i,n}+3!\,l^3D^{m+3}b_{i,n}+\text{etc.}}^{\text{For } a},$$

which enables us to pass readily from the coefficients calculated for a to those for $a+\Delta a$.

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26*th March* 1909.

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Notes added by Frank Robbins.

The two equations for $b_{0,1}$ may be derived from equations (4) and (7) respectively as given by Verhulst (*loc. cit.*) if it is remembered that his D is the complete elliptic integral of the first class

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - c^2 \sin^2 \phi}}$$

and his b is the complementary modulus $\sqrt{1 - c^2}$.

The second and more accurate formula may be slightly simplified—it easily becomes

$$b_{0,1} = \frac{32}{\left\{1 + a_1^{\frac{1}{2}} + \sqrt[4]{8\left(a_1^{\frac{1}{2}} + a_1^{\frac{3}{2}}\right)}\right\}^2} \left\{1 + 2q^{16} + 2q^{64} + \ldots\ldots\right\}^2 ;$$

but even so, it is somewhat troublesome, and the first formula would probably be adopted by most computers. It is certainly quicker even if a special value of N_1 is computed instead of being interpolated from the table, page 649.

Computation of N_1.

The exact value of N_1 is $\log_{10}(1 + 2q^4 + 2q^{16} + 2q^{36} + \ldots\ldots)^2$
or $\log_{10}(1 + 4q^4 + 4q^8 + \ldots\ldots)$.

If a is not greater than 0·85 it will be sufficient to stop at $4q^4$, and then it is easy to see from the equations

$$a^2 + a_1^2 = 1 \qquad \frac{l}{2} = \frac{1 - a_1^{\frac{1}{2}}}{2\left(1 + a_1^{\frac{1}{2}}\right)}$$

$$q = \frac{l}{2} + 2\left(\frac{l}{2}\right)^5 + 15\left(\frac{l}{2}\right)^9 + \ldots\ldots$$

that $N_1 = m \log_e (1 + 4q^4 + 4q^8 + \ldots)$
or $N_1 = 4mq^4$ very nearly

$$N_1 = 4m\left\{\left(\frac{l}{2}\right)^4 + 8\left(\frac{l}{2}\right)^8\right\} \qquad m \text{ being } 0{\cdot}4342945.$$

This is a very convenient working formula, and it can be computed for any given value of a in less than ten minutes. Zech's addition table serves admirably here; the argument being the *difference of the logarithms* of the two terms within the brackets. This *difference* can be found mentally from the log of the first term $\left(\frac{l}{2}\right)^4$ without the log $8\left(\frac{l}{2}\right)^8$ being known. It is merely necessary to add 0·903 0900 to the mantissa of log $\left(\frac{l}{2}\right)^4$ and to